

SYSTEMS OF PARAMETERS AND THE COHEN-MACAULAY PROPERTY

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ABSTRACT. Let R be a commutative, Noetherian, local ring and M an R -module. Consider the module of homomorphisms $\text{Hom}_R(R/\mathfrak{a}, M/\mathfrak{b}M)$ where $\mathfrak{b} \subseteq \mathfrak{a}$ are parameter ideals of M . When $M = R$ and R is Cohen-Macaulay, Rees showed that this module of homomorphisms is always isomorphic to R/\mathfrak{a} , and in particular, a free module over R/\mathfrak{a} of rank one. In this work, we study the structure of such modules of homomorphisms for general M .

1. INTRODUCTION

Let R be a commutative, Noetherian, local ring. This work concerns the module of homomorphisms $\text{Hom}_R(R/\mathfrak{a}, R/\mathfrak{b})$ where \mathfrak{a} and \mathfrak{b} are parameter ideals of R with $\mathfrak{b} \subseteq \mathfrak{a}$.

An immediate consequence of a result of Rees [4] is that when R is Cohen-Macaulay, this module of homomorphisms is isomorphic to R/\mathfrak{a} . In particular, as an R/\mathfrak{a} -module, it is free of rank one. The focus of this work is to study the structure of this module of homomorphisms when R is not Cohen-Macaulay. Our main results identify circumstances under which it is decomposable and not free.

When R has dimension one and depth zero and \mathfrak{a} and \mathfrak{b} are in sufficiently high powers of the maximal ideal, we prove that $\text{Hom}_R(R/\mathfrak{a}, R/\mathfrak{b})$ is neither indecomposable nor free as an R/\mathfrak{a} -module.

We can extend the result about decomposability both to modules and to higher dimensions. In particular, for M a nonzero, finitely generated R -module we consider the module $\text{Hom}_R(R/\mathfrak{a}, M/\mathfrak{b}M)$ where $\mathfrak{b} \subseteq \mathfrak{a}$ are parameter ideals of M . When M is not Cohen-Macaulay, we can show that the module of homomorphisms decomposes for any parameter ideal \mathfrak{a} and for \mathfrak{b} chosen to be generated by suitable powers of any system of parameters generating \mathfrak{a} . This result generalizes recent work of K. Bahmanpour and R. Naghipour [1] in the case where $M = R$. Specifically, when R is not Cohen-Macaulay, they showed there exist some parameter ideals $\mathfrak{b} \subseteq \mathfrak{a}$ of R for which $\text{Hom}_R(R/\mathfrak{a}, R/\mathfrak{b})$ is not cyclic.

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2. PRELIMINARIES

Throughout, R will be a commutative, Noetherian, local ring with maximal ideal \mathfrak{m} , and M will be a finitely generated R -module of dimension d . A *system of parameters* of M is a set of d elements generating an ideal \mathfrak{a} such that $M/\mathfrak{a}M$ has finite length. An ideal \mathfrak{a} generated by a system of parameters is called a *parameter ideal*. We begin by reviewing the consequence of Rees' result in the Cohen-Macaulay case.

Remark 2.1. If M is a Cohen-Macaulay R -module of dimension d , and $\mathfrak{b} \subseteq \mathfrak{a}$ are parameter ideals of M , then

$$\mathrm{Hom}_R(R/\mathfrak{a}, M/\mathfrak{b}M) \cong M/\mathfrak{a}M.$$

In particular, when $M = R$, this is a free R/\mathfrak{a} -module of rank one and hence indecomposable.

Indeed, the elements of a system of parameters of M form an M -regular sequence. The isomorphism above is deduced from Rees' Theorem [2, Lemma 1.2.4]:

$$\begin{aligned} \mathrm{Hom}_R(R/\mathfrak{a}, M/\mathfrak{b}M) &\cong \mathrm{Ext}_R^d(R/\mathfrak{a}, M) \\ &\cong \mathrm{Hom}_R(R/\mathfrak{a}, M/\mathfrak{a}M) \\ &\cong M/\mathfrak{a}M. \end{aligned}$$

We now recall some well-known results.

Fact 2.2. Let $\mathfrak{a} = a_1, \dots, a_d \in \mathfrak{m}$ and $\bar{\mathfrak{a}} = \bar{a}_1, \dots, \bar{a}_d$ be the images of the a_i in $R/\mathrm{ann}(M)$. Then \mathfrak{a} is a system of parameters of M if and only if $\bar{\mathfrak{a}}$ is a system of parameters of the ring $S := R/\mathrm{ann}(M)$.

The integer n appearing in the next result plays a key role in the main results. We include a proof for completeness.

Lemma 2.3. *Let (R, \mathfrak{m}) be a local Noetherian ring and M a finitely generated R -module. There exists an integer n such that $\mathfrak{m}^n M \cap \Gamma_{\mathfrak{m}}(M) = (0)$.*

Proof. Since $\Gamma_{\mathfrak{m}}(M)$ is Artinian, the descending chain of submodules

$$(\mathfrak{m}M \cap \Gamma_{\mathfrak{m}}(M)) \supseteq (\mathfrak{m}^2M \cap \Gamma_{\mathfrak{m}}(M)) \supseteq \dots$$

must stabilize. That is, there is some $n \in \mathbb{N}$ such that

$$\mathfrak{m}^{n+i}M \cap \Gamma_{\mathfrak{m}}(M) = \mathfrak{m}^nM \cap \Gamma_{\mathfrak{m}}(M)$$

for all integers $i \geq 0$. Thus

$$\begin{aligned} \mathfrak{m}^nM \cap \Gamma_{\mathfrak{m}}(M) &= \bigcap_{i \geq n} (\mathfrak{m}^iM \cap \Gamma_{\mathfrak{m}}(M)) \\ &\subseteq \bigcap_{i \geq n} \mathfrak{m}^iM \\ &= (0) \end{aligned}$$

by the Krull Intersection Theorem. □

Remark 2.4. In fact, given any finite-length submodule $L \subseteq M$, we have $\mathfrak{m}^n M \cap L = (0)$ where n is the integer of Lemma 2.3.

The next result is in the spirit of [3, Prop 4.7.13]. We include a proof in order to obtain specific bounds on the powers of a in this special case.

Proposition 2.5. *Let A be any commutative ring, L an A -module, and $a, b \in A$. Then, for arbitrary positive integers $p \leq q \leq r$ we have the equality*

$$(ba^r L : a^p) = a^{r-q}(ba^q L : a^p) + (0 :_L a^p).$$

Proof. First let $x \in (ba^r L : a^p)$. Then $a^p x = ba^r y$ for some y in L . Now $a^p(x - ba^{r-p}y) = 0$ so that $x - ba^{r-p}y \in (0 :_L a^p)$. Moreover

$$ba^{r-p}y = a^{r-q} \cdot ba^q y \in a^{r-q}(ba^q L : a^p).$$

We now have

$$x = ba^{r-p}y + (x - ba^{r-p}y) \in a^{r-q}(ba^q L : a^p) + (0 :_L a^p).$$

For the other inclusion, let $x \in a^{r-q}(ba^q L : a^p) + (0 :_L a^p)$ and write $x = a^{r-q}y + z$ with y in $(ba^q L : a^p)$ and z in $(0 :_L a^p)$. We can write $a^p y = ba^q w$ for some w in L . Thus we may rewrite $a^p x$ as

$$\begin{aligned} a^p x &= a^p(a^{r-q}y + z) \\ &= a^{r-q}ba^q w + a^p z \\ &= ba^r w, \end{aligned}$$

which is in $ba^r L$. Thus $x \in (ba^r L : a^p)$ as desired. \square

We will use the next result in Sections 3 and 4.

Lemma 2.6. *Let R be a local Noetherian ring, $J \subseteq I$ ideals of R with $\sqrt{I} = \sqrt{J}$, and N a nonzero finitely generated R -module. If $\text{Hom}_R(R/J, N)$ is decomposable, then so is $\text{Hom}_R(R/I, N)$.*

Proof. Suppose that $\text{Hom}_R(R/J, N) = X \oplus Y$ where X and Y are nonzero R -modules. There are isomorphisms

$$\begin{aligned} \text{Hom}_R(R/I, N) &\cong \text{Hom}_R((R/I) \otimes_R (R/J), N) \\ &\cong \text{Hom}_R(R/I, \text{Hom}_R(R/J, N)) \\ &\cong \text{Hom}_R(R/I, X) \oplus \text{Hom}_R(R/I, Y). \end{aligned}$$

By symmetry, it suffices to show that $\text{Hom}_R(R/I, X) \neq 0$. It is clear that $JX = (0)$ since $X \subseteq \text{Hom}_R(R/J, N)$. Choose $P \in \text{Ass}_R X$ such that $J \subseteq P$. Since $\sqrt{J} = \sqrt{I}$, one has $I \subseteq P$, so there are maps

$$R/I \twoheadrightarrow R/P \hookrightarrow X.$$

The composition of these maps is nonzero, and so $\text{Hom}_R(R/I, X) \neq (0)$ as desired. \square

This final result will be used in Section 4.

Lemma 2.7. *Let R be a local Noetherian ring and M a R -module of dimension $d \geq 2$. If M is not Cohen-Macaulay, then for any system of parameters a_1, \dots, a_d of M , there exist positive integers i and s such that $M/a_i^s M$ is not Cohen-Macaulay.*

Proof. If some a_i is M -regular, then $M/a_i M$ is not Cohen-Macaulay, so we may assume that each a_i is a zero-divisor on M . Suppose, by way of contradiction that $M/a_1^s M$ is Cohen-Macaulay for each $s \geq 1$. Then a_2, \dots, a_d is a regular sequence on $M/a_1^s M$ for all integers $s \geq 1$. In particular a_2 is $M/a_1^s M$ -regular for all integers $s \geq 1$. We claim this implies a_2 is M -regular, which is a contradiction. Indeed, suppose $a_2 m = 0$ for some $m \in M$. Then $a_2 \bar{m} = 0$ in $M/a_1^s M$ for all integers $s \geq 1$ so that $m \in a_1^s M$ for all integers $s \geq 1$. By the Krull Intersection Theorem, we have $m = 0$ implying a_2 is M -regular, a contradiction. \square

3. DIMENSION ONE

We start with results on modules of dimension one and depth zero since we are able to obtain stronger results in this case. We show $\text{Hom}_R(R/\mathfrak{a}, M/\mathfrak{b}M)$ will decompose if the parameter ideal \mathfrak{b} is chosen to be in a sufficiently high power of the maximal ideal.

Theorem 3.1. *Let (R, \mathfrak{m}) be a local Noetherian ring, M a nonzero finitely generated R -module of dimension one and depth zero, and n an integer such that $\mathfrak{m}^n M \cap \Gamma_{\mathfrak{m}}(M) = (0)$. For any parameter a of M , and any parameter b of M with $b \in (a^{n+1})$, the following R -module is decomposable:*

$$\text{Hom}_R(R/(a), M/bM).$$

Remark 3.2. The integer n in the statement exists by Lemma 2.3. Note that $n \geq 1$ because $\Gamma_{\mathfrak{m}}(M) \neq (0)$.

Proof of Theorem 3.1. Set $S := R/\text{ann}(M)$ and let $(\bar{})$ denote the image in S . In light of Fact 2.2, \bar{a}, \bar{b} are parameters of S . Moreover there is an R -module isomorphism

$$\text{Hom}_S(S/(\bar{a}), M/\bar{b}M) \cong \text{Hom}_R(R/(a), M/bM).$$

Thus, by replacing R with S , we may assume that M is faithful as an R -module.

Write $b = ca^{n+1}$. Since M is faithful, we have $\sqrt{(a)} = \mathfrak{m}$ and so

$$(0 :_M a) \subseteq \Gamma_{(a)}(M) = \Gamma_{\mathfrak{m}}(M).$$

Thus we know

$$(1) \quad (0 :_M a) \cap ca^n M \subseteq \Gamma_{\mathfrak{m}}(M) \cap \mathfrak{m}^n M = (0).$$

By Proposition 2.5 we have

$$(2) \quad (ca^{n+1}M : a) = a^n(caM : a) + (0 :_M a).$$

We now claim that

$$a^n(caM : a) = ca^n M.$$

Indeed, it is clear that elements of $ca^n M$ are also elements of $a^n(caM : a)$. For the reverse inclusion, let $x \in a^n(caM : a)$ and write $x = a^n m$ for some $m \in (caM : a)$. So we have $am = cam'$ for some $m' \in M$. Then

$$x = a^n m = a^{n-1} \cdot am = ca^n m' \in ca^n M.$$

Equation (2) now becomes

$$(3) \quad (ca^{n+1}M : a) = ca^n M + (0 :_M a).$$

Next we want to show

$$(4) \quad ca^{n+1}M = ca^n M \cap [(0 :_M a) + ca^{n+1}M].$$

Elements in $ca^{n+1}M$ also live in both $ca^n M$ and $(0 :_M a) + ca^{n+1}M$. For the other inclusion, let $x \in ca^n M \cap [(0 :_M a) + ca^{n+1}M]$ and write

$$x = ca^n m = \eta + ca^{n+1}m'$$

for some $m, m' \in M$, and $\eta \in (0 :_M a)$. Then

$$\begin{aligned} \eta &= ca^n m - ca^{n+1}m' \\ &\in (0 :_M a) \cap ca^n M = (0) \end{aligned} \quad \text{by (1).}$$

Equation (4) follows. Now there are isomorphisms

$$\begin{aligned} \text{Hom}_R(R/(a), M/bM) &\cong \frac{(bM : a)}{bM} \\ &\cong \frac{ca^n M + (0 :_M a)}{ca^{n+1}M} && \text{by (3)} \\ &\cong \frac{ca^n M}{ca^{n+1}M} \oplus \frac{(0 :_M a) + ca^{n+1}M}{ca^{n+1}M} && \text{by (4).} \end{aligned}$$

All that remains to prove is that both summands are nonzero.

If the summand on the left were zero, then $ca^n M = (0)$ by Nakayama's Lemma, a contradiction as $ca^{n+1} = b$ is a parameter of M .

If the summand on the right were zero, then

$$(0 :_M a) \subseteq ca^{n+1}M.$$

By Equation (1) we have

$$(0 :_M a) = (0 :_M a) \cap ca^{n+1}M = (0).$$

This is also a contradiction as $\text{depth}_R M = 0$. Thus $\text{Hom}_R(R/(a), M/bM)$ is decomposable, as desired. \square

When R is a Cohen-Macaulay ring, we know from Remark 2.1 that the R/\mathfrak{a} -module $\text{Hom}_R(R/\mathfrak{a}, R/\mathfrak{b}) \cong R/\mathfrak{a}$ is not only indecomposable, but also free. When R is one dimensional and not Cohen-Macaulay, we can prove that in addition to decomposing, this module will be non-free if the parameters are chosen to be in sufficiently high powers of the maximal ideal.

Theorem 3.3. *Let (R, \mathfrak{m}) be a Noetherian local ring of dimension one and depth zero, and n an integer such that $\mathfrak{m}^n \cap \Gamma_{\mathfrak{m}}(R) = (0)$. For any parameter a in \mathfrak{m}^n and any parameter b in (a^2) , the $R/(a)$ -module*

$$\mathrm{Hom}_R(R/(a), R/(b))$$

is decomposable and has a non-free summand.

Remark 3.4. Again, the integer n in the statement exists by Lemma 2.3 and must be positive since $\Gamma_{\mathfrak{m}}(R) \neq (0)$.

Proof of Theorem 3.3. We will first prove that the module decomposes. Both the proof of this fact and the decomposition obtained are similar to those found in the proof of Theorem 3.1. Write $I = \Gamma_{\mathfrak{m}}(R)$. For any $x \in \mathfrak{m}^n$, we know $(x) \cap I = (0)$ and hence $xI = (0)$. If $x \in \mathfrak{m}^n$ is also a parameter, then we know $\Gamma_{(x)}(R) = I$, and $(0 : x) = I$ as well. Indeed, $\sqrt{(x)} = \mathfrak{m}$ and since $xI = 0$ we have

$$I \subseteq (0 : x) \subseteq \Gamma_{(x)}(R) = I.$$

Let $a \in \mathfrak{m}^n$ and $b \in (a^2)$ be parameters and write $b = ca^2$. Applying Proposition 2.5 with $p = q = 1$ and $r = 2$, we obtain the equality

$$(5) \quad ((ca^2) : a) = a((ca) : a) + (0 : a).$$

We now note that $a((ca) : a) = (ca)$. We may thus rewrite Equation (5) as

$$(6) \quad ((b) : a) = (ca) + I.$$

Next we wish to show that

$$(7) \quad (ca^2) = (ca) \cap [I + (ca^2)].$$

The inclusion \subseteq is clear. For the other inclusion, let $x \in (ca) \cap [I + (ca^2)]$ and write $x = rca = \eta + r'ca^2$ for some $r, r' \in R$ and $\eta \in I$. Then

$$\begin{aligned} \eta &= rca - r'ca^2 \\ &\in (a) \cap I \\ &\subseteq \mathfrak{m}^n \cap I = (0). \end{aligned}$$

Hence $\eta = 0$ and $x = r'ca^2 \in (a^2)$. Hence there are isomorphisms of $R/(a)$ -modules

$$\begin{aligned} \mathrm{Hom}_R(R/(a), R/(b)) &\cong \frac{((b) : a)}{(b)} \\ &\cong \frac{(ca) + I}{(ca^2)} && \text{by (6)} \\ &\cong \frac{(ca)}{(ca^2)} \oplus \frac{I + (ca^2)}{(ca^2)} && \text{by (7)}. \end{aligned}$$

Next we show that both summands are nonzero.

If the summand on the left were zero, then Nakayama's Lemma implies $ca = 0$, a contradiction as $ca^2 = b$ is a parameter and hence nonzero.

If the summand on the right were zero, then $I \subseteq (ca^2)$ so that

$$I = I \cap (ca^2) \subseteq I \cap (a) = (0),$$

a contradiction as the depth of R is zero.

We now show that the summand on the left, that is $(ca)/(ca^2)$, is not a free $R/(a)$ -module. To that end, recall that $I \cap (a) = (0)$, but $I \neq (0)$ so we can choose an element $y \in I \setminus (a)$. Since $aI = (0)$, \bar{y} is a nonzero element of $R/(a)$ that annihilates $(ca)/(ca^2)$, and hence $(ca)/(ca^2)$ cannot be free as an $R/(a)$ -module. \square

4. HIGHER DIMENSIONS

In higher dimensions, we can also prove a decomposition theorem. However, Example 5.4 shows that Theorem 3.1 is not strong enough to use the induction technique in Theorem 4.1 to prove there is an integer N such that $\text{Hom}_R(R/\mathfrak{a}, R/(a_1^{n_1}, \dots, a_d^{n_d}))$ decomposes for all $n_i \geq N$. K. Bahmanpour and R. Naghipour [1] prove that when R is not Cohen-Macaulay $\text{Hom}_R(R/\mathfrak{a}, R/\mathfrak{b})$ is not cyclic for some parameter ideals \mathfrak{a} and \mathfrak{b} with $\mathfrak{b} \subseteq \mathfrak{a}$.

Theorem 4.1. *Let R be a local Noetherian ring and M a finitely generated R -module of dimension d . If M is not Cohen-Macaulay, then, for any system of parameters $\mathbf{a} = a_1, \dots, a_d$ of M , there exist integers $n_1, \dots, n_d \in \mathbb{N}$ such that the following R -module is decomposable:*

$$\text{Hom}_R(R/(\mathbf{a}), M/(a_1^{n_1}, \dots, a_d^{n_d})M).$$

Proof. As in the proof of Theorem 3.1, we may reduce to the case that M is a faithful module. We proceed by induction on d , the case $d = 1$ being covered by Theorem 3.1.

Assume, now, that $d \geq 2$. By Lemma 2.7, we can find some $i \leq d$ and a positive integer n_i such that $M/a_i^{n_i}M$ is not Cohen-Macaulay. We may harmlessly assume $i = 1$. Set

$$\bar{R} := R/(a_1^{n_1}) \quad \text{and} \quad \bar{\mathbf{a}} := (\bar{a}_2, \dots, \bar{a}_d).$$

Then $\bar{\mathbf{a}}$ is a parameter ideal of \bar{R} . Since \bar{R} has dimension $d-1$, by induction there are natural numbers n_2, \dots, n_d such that the \bar{R} -module

$$U := \text{Hom}_{\bar{R}}(\bar{R}/\bar{\mathbf{a}}, \bar{M}/(\bar{a}_2^{n_2}, \dots, \bar{a}_d^{n_d})\bar{M})$$

decomposes. Since there is an isomorphism

$$U \cong \text{Hom}_R(R/(a_1^{n_1}, a_2, \dots, a_d), M/(a_1, a_2^{n_2}, \dots, a_d^{n_d})M),$$

then applying Lemma 2.6, gives the desired decomposition. \square

The result below is a version of Theorem 3.3 for rings of arbitrary dimension.

Theorem 4.2. *Let R be a local Noetherian ring of dimension d . If R is not Cohen-Macaulay, then for any system of parameters a_1, \dots, a_d of R , there*

exist integers $n_1, \dots, n_d, N_1, \dots, N_d$ with $n_i \leq N_i$ for $i = 1, \dots, d$ such that the $R/(a_1^{n_1}, \dots, a_d^{n_d})$ -module

$$\mathrm{Hom}_R(R/(a_1^{n_1}, \dots, a_d^{n_d}), R/(a_1^{N_1}, \dots, a_d^{N_d}))$$

is decomposable and not free.

Proof. We proceed by induction on d . If $d = 1$, then choosing n_1 to be the n from Theorem 3.3 and $N_1 = 2n_1$ works.

Now suppose that $d \geq 2$. By Lemma 2.7, we can find integers i and n_i such that $R/(a_i^{n_i})$ is not Cohen-Macaulay. We may harmlessly assume $i = 1$. Set $S := R/(a_1^{n_1})$ let $(-)$ denote the image in S . Then $\overline{a_2}, \dots, \overline{a_d}$ is a system of parameters of S and, by induction, there exist integers $n_2, \dots, n_d, N_2, \dots, N_d$ such that the $S/(\overline{a_2}^{n_2}, \dots, \overline{a_d}^{n_d})$ -module

$$U := \mathrm{Hom}_S(S/(\overline{a_2}^{n_2}, \dots, \overline{a_d}^{n_d}), S/(\overline{a_2}^{N_2}, \dots, \overline{a_d}^{N_d}))$$

decomposes and has a non-free summand. Note that

$$S/(\overline{a_2}^{n_2}, \dots, \overline{a_d}^{n_d}) \cong R/(a_1^{n_1}, \dots, a_d^{n_d}).$$

Setting $N_1 = n_1$ we then have

$$U \cong \mathrm{Hom}_R(R/(a_1^{n_1}, \dots, a_d^{n_d}), R/(a_1^{N_1}, \dots, a_d^{N_d}))$$

and this gives the desired decomposition and non-free summand. \square

5. EXAMPLES

In this section, we focus on examples. In particular, we investigate the structure of the R/\mathfrak{a} -module $\mathrm{Hom}_R(R/\mathfrak{a}, R/\mathfrak{b})$ for concrete examples of R , \mathfrak{a} , and \mathfrak{b} .

Let us take $M = R$ in Theorem 4.1. If we take $n_i = 1$ for each i then

$$(8) \quad \mathrm{Hom}_R(R/(a_1, \dots, a_d), R/(a_1^{n_1}, \dots, a_d^{n_d})) \cong R/(a_1, \dots, a_d)$$

is a free $R/(a_1, \dots, a_d)$ -module of rank one. Our first example shows that Equation 8 sometimes holds even when R is not Cohen-Macaulay and at least one of the n_i 's is greater than one.

Example 5.1. Consider the parameter y of $R = k[[x, y]]/(x^2, xy^2)$. Then we have

$$\mathrm{Hom}_R(R/(y), R/(y^2)) \cong \frac{(y^2) :_R y}{(y^2)} = \frac{(y)}{(y^2)} \cong R/(y).$$

The next example shows that the module $\mathrm{Hom}_R(R/\mathfrak{a}, R/\mathfrak{b})$ can be neither cyclic nor decomposable and also that the bound in Theorem 3.1 is not always optimal.

Example 5.2. Consider the parameter y^2 of $R = k[[x, y]]/(x^2, xy^m)$. Then

$$U_t := \mathrm{Hom}_R(R/(y^2), R/(y^{2t}))$$

is

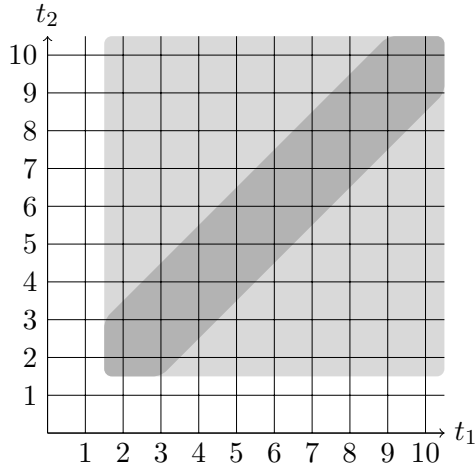
$$\begin{cases} \text{cyclic,} & \text{if } t < \frac{m+1}{2}, \\ \text{indecomposable, but not cyclic,} & \text{if } t = \frac{m+1}{2}, \text{ and} \\ \text{decomposable,} & \text{if } t > \frac{m+1}{2}. \end{cases}$$

However, Theorem 3.1 only predicts that U_t decomposes for $t > m + 1$ since $\mathfrak{m}^n \cap \Gamma_{\mathfrak{m}}(R) \neq 0$ for $n < m$.

The next example shows that even when all of the parameters are zero divisors, M may have positive depth, and M/aM may be Cohen-Macaulay.

Example 5.3. Consider the ring $R = k[[x, y, z]]/(x^2, xyz)$ of dimension two and depth one along with the system of parameters y, z . Both y and z are zero-divisors in R and both $R/(y)$ and $R/(z)$ are Cohen-Macaulay rings of dimension one.

FIGURE 1. In this figure, a lattice point (t_1, t_2) corresponds to the module $\text{Hom}_R(R/(y, z), R/(y^{t_1}, z^{t_2}))$ from Example 5.4. The modules corresponding to lattice points in the light grey regions are known to decompose due to Theorem 3.1. The modules corresponding to lattice points in the middle dark grey region are known to decompose by direct computation. The modules corresponding to lattice points where $t_1 = 1$ or $t_2 = 1$ are indecomposable since $R/(y)$ and $R/(z)$ are both Cohen-Macaulay rings.



Theorems 3.1 and 3.3, which give bounds on the powers needed to make $\text{Hom}_R(R/\mathfrak{a}, M/\mathfrak{b}M)$ decompose and be non-free, apply only in dimension one. However, examples seem to indicate that the R/\mathfrak{a} -module

$$\text{Hom}_R(R/\mathfrak{a}, R/(a_1^{n_1}, \dots, a_d^{n_d}))$$

is neither free nor indecomposable if the n_i are large enough. One such example is explained below.

Example 5.4. Again, consider the ring $R = k[[x, y, z]]/(x^2, xyz)$ of dimension two and depth one along with the system of parameters y, z . If $n_1 \geq 2$, then $S_{n_1} := R/(y^{n_1})$ is not Cohen-Macaulay (since the non-zero element xy^{n_1-1} is in the socle). Letting \mathfrak{m} be the maximal ideal of S_{n_1} we have $\mathfrak{m}^n \cap \Gamma_{\mathfrak{m}}(S_{n_1}) = 0$ if and only if $n \geq n_1 + 2$. By symmetry, the same holds for the ring $T_{n_2} := R/(z^{n_2})$. Thus Theorem 3.1 gives that $U_{n_1, n_2} := \text{Hom}_R(R/(y, z), R/(y^{n_1}, z^{n_2}))$ decomposes for all $n_1, n_2 \geq 2$ with $|n_1 - n_2| > 2$. However, direct computation shows that U_{n_1, n_2} actually decomposes for all $n_1, n_2 \geq 2$. See Figure 1 for a visual representation of this.

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